The Geometry of Weak Fields: Curvature

Introduction

In this talk, we will begin to think about curved spaces, and we will consider these questions:

1. What is the difference between *accelerated* motion in a space that is *flat* vs. *non-accelerated* motion in a space that is *curved*?
2. What do we mean by "accelerated motion"?
3. What do we even mean by "a space that is curved?"

Acceleration

- Straightforwardly: If we observe a velocity vector $\vec{v}$ changing with time, we (by definition) observe an acceleration.
- But, what is the cause of the acceleration?
- Newton said, "a force!"
- Source of the force?
- What if our space is curved? How would that affect $\vec{v}$?

A Familiar Curvilinear Geometry

To start, lets think about the curvature that we are most used to: that of a spherical surface.

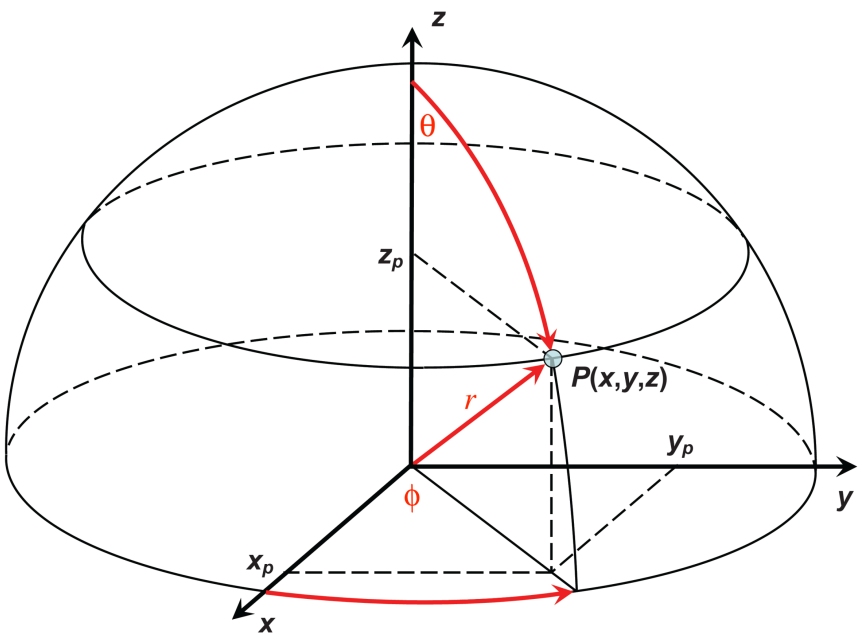

Figure: The spherical coordinate system of physics. Note that, in our context, the radius of curvature of the sphere r is not a coordinate; it merely serves as a parameter to scale the sphere whose *surface* comprises our space.

The Distance Interval

We know that the distance between any two points on the surface of a sphere is related to the distance interval on the surface:

$$ds^2 = (rd\theta)^2 + (r\sin\theta d\phi)^2 \tag{1}$$

$$= r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{2}$$

where θ and ϕ are the polar and azimuthal angles, respectively (see Fig. 1), and r is a scaling parameter–*not* a coordinate.

Major Topic: Distance Interval and the Metric Tensor

We can write the square of the distance element on the surface as a product of the coordinate distances and the metric tensor:

$$\begin{aligned} ds^2 &= g_{ij}\, dx^i\, dx^j \\ &= g_{11}\, dx^1\, dx^1 + g_{12}\, dx^1\, dx^2 + g_{21}\, dx^2\, dx^1 + g_{22}\, dx^2\, dx^2 \\ &= g_{11}\, d\theta\, d\theta + g_{12}\, d\theta\, d\phi + g_{21}\, d\phi\, d\theta + g_{22}\, d\phi\, d\phi \\ &= g_{11}\, d\theta^2 + g_{12}\, d\theta\, d\phi + g_{21}\, d\phi\, d\theta + g_{22}\, d\phi^2 \end{aligned} \tag{3}$$

Distance Interval and the Metric Tensor (cont.)

Comparing

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

with

$$ds^2 = g_{11} d\theta^2 + g_{12} d\theta d\phi + g_{21} d\phi d\theta + g_{22} d\phi^2$$

we see that

$$\begin{aligned} g_{11} &= r^2 \\ g_{22} &= r^2 \sin^2\theta \\ g_{12} &= g_{21} = 0 \end{aligned}$$

Metric Tensor on A Sphere

and therefore

$$g = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \sin^2 \theta \end{pmatrix} \tag{4}$$

- The thing to notice about this metric is that some (well, one) of its elements depends on position; in particular, the g_{22} term depends on θ. (Remember, r in this space is a scaling parameter, not a coordinate.)
- When the metric of a space depends on the position, that *can* be a signature of curvature.
- But–having a variable metric is *not* enough to say that a space is curved, only that the *coordinates* that we are using are curved.

We'll see an example of this in a moment.

Coordinates or Space?

- You might argue that these spherical surface coordinates are not actually describing a curved space; they are describing a curved 2-dimensional *surface* in an otherwise *flat* 3-dimensional space.
- However, if we consider only the curved surface itself, then the metric is describing everything about the geometry of that space. We don't have to appeal to a higher-dimensional space in order to make sense of what the metric tells us.
- And we don't have to think of the radius r as anything other than a parameter that tells us how big things are in our space.

Space or Coordinates?

- In other words: We are used to thinking of r as the actual radius of an actual sphere, but it need not be.
- We will find that the same is true in general relativity when we consider the curvature of 3-space or 3+1 space: we *can* imagine those spaces embedded in higher dimensional spaces if we want to, but it is really no help to do so. We cannot visualize or measure any higher dimension spaces anyway, and so any such putative space is of no use to us, either conceptually, mathematically, or physically. –And in any event,
- **the metric will tell us everything we need to know about the curvature of the spaces in which we make our measurements**.

Metric Tensor in Oblique Space

As another example, consider a flat Euclidian space as shown in Fig.

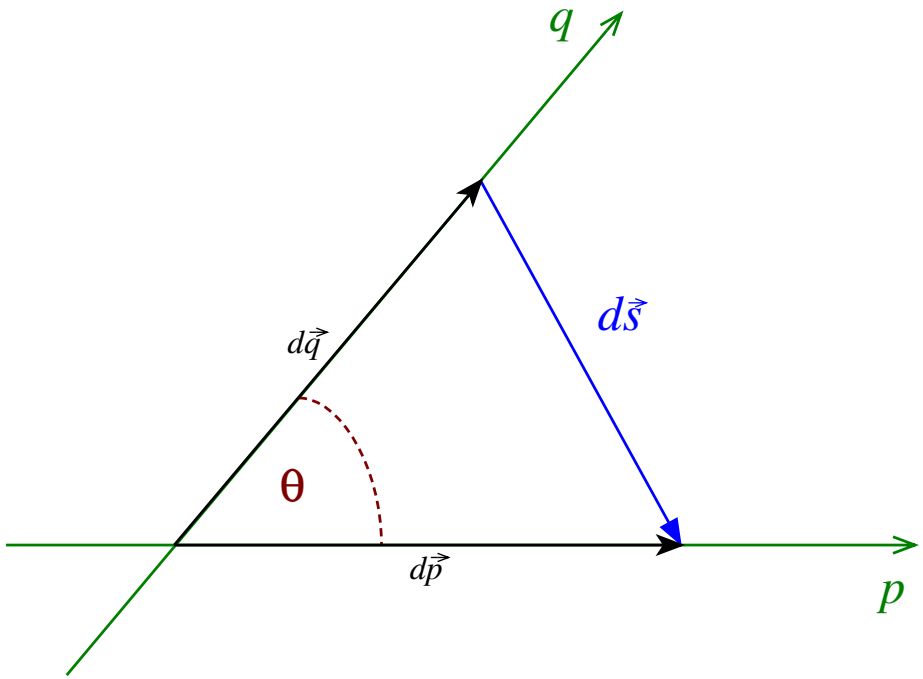

Figure: In an oblique coordinate system, in which the two coordinate axes $x^1 = p$ and $x^2 = q$ are not perpendicular, the distance interval depends on a cross term between the two coordinates.

Metric Tensor in Oblique Space (cont.)

The length interval squared is

$$ds^2 = dp^2 + dq^2 - 2dpdq\cos\theta \tag{5}$$

Once again we will write the square of the distance element on the surface as a product of the coordinate distances and the metric tensor:

$$\begin{aligned} ds^2 &= g_{ij}\,dx^i\,dx^j \\ &= g_{11}\,dx^1\,dx^1 + g_{12}\,dx^1\,dx^2 + g_{21}\,dx^2\,dx^1 + g_{22}\,dx^2\,dx^2 \\ &= g_{11}\,dpdp + g_{12}\,dpdq + g_{21}\,dqdp + g_{22}\,dqdq \\ &= g_{11}\,dp^2 + g_{12}\,dpdq + g_{21}\,dqdp + g_{22}\,dq^2 \end{aligned} \tag{6}$$

Equating,

$$g_{11}\,dp^2 + g_{12}\,dpdq + g_{21}\,dqdp + g_{22}\,dq^2 = dp^2 + dq^2 - 2dpdq\cos\theta$$

Metric Tensor in Oblique Space (cont.)

And we therefore obtain

$$
\begin{aligned}
g_{11} &= 1 \\
g_{12} &= -\cos\theta \\
g_{21} &= -\cos\theta \\
g_{22} &= 1
\end{aligned}
$$

Or, more simply

$$
g = \begin{pmatrix} 1 & -\cos\theta \\ -\cos\theta & 1 \end{pmatrix}
$$

Metric Tensor in Oblique Space (cont.)

- This is usually presented as a way to compute the third side of a triangle when two sides, along with the angle between them, are already known (i.e., the law of cosines).
- But it can also be thought of as the square of the distance element for an oblique coordinate system such as the one shown in Fig. 2. In that system the angle θ is not one of the coordinates, it is merely a parameter that tells us the amount by which the coordinate axes are skewed.
- In this regard it plays a role similar to that of the radius r in the spherical surface coordinates we described previously.

(There are systems where these sorts of oblique coordinates are convenient, by the way. An example would be to use them to describe the oblique orientations of some crystals. In those cases the individual p and q coordinate axes do not even have to have the same scale since the atomic spacings along the different directions can be different.)

Plane-Polar Coordinates

Our next example: Calculate the metric tensor for polar coordinates.

Expressing the (squared) distance interval in terms of the metric,

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{7}$$

The differential line elements in plane-polar coordinates are

$$\begin{aligned} ds_r &= dr\left(= dx^1\right) \\ ds_\phi &= r d\phi\left(= dx^2\right) \end{aligned} \tag{8}$$

The (squared) distance element in a plane is given by

$$ds^2 = d\vec{s}\cdot d\vec{s} = dx^i\, dx^i = dr^2 + r^2 d\phi^2$$

(Notice that r is *not* a parameter here; this time, it *is* a coordinate of the space.)

Plane-Polar Coordinates (cont.)

Expanding Eq. 7 for $\alpha, \beta = 1, 2$

$$ds^2 = g_{11}\,dx^1\,dx^1 + g_{12}\,dx^1\,dx^2 + g_{21}\,dx^2\,dx^1 + g_{22}\,dx^2\,dx^2$$

Equating the expressions for ds^2, we get

$$g_{11}\,dx^1\,dx^1 + g_{12}\,dx^1\,dx^2 + g_{21}\,dx^2\,dx^1 + g_{22}\,dx^2\,dx^2 = dr^2 + r^2\,d\phi^2$$

Having defined $dx^1 \equiv dr$ and $dx^2 \equiv r d\phi$, this becomes

$$g_{11}\,dx^1\,dx^1 + g_{12}\,dx^1\,dx^2 + g_{21}\,dx^2\,dx^1 + g_{22}\,dx^2\,dx^2 = dx^1\,dx^1 + r^2\,dx^2\,dx^2$$

Plane-Polar Coordinates (cont.)

whence

$$
\begin{aligned}
g_{11} &= 1 \\
g_{12} &= 0 \\
g_{21} &= 0 \\
g_{22} &= r^2
\end{aligned}
$$

or, in matrix form,

$$
g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{9}
$$

Something is odd here!

Metrics are not Unique

This metric depends on position; and yet, it describes a *flat* space. We have two different metrics that describe exactly the same space, which in this case is flat:

$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$
Cartesian	Polar

One of them has constant metric coefficients, the other (like the metric for our 2d spherical surface), does not.

Metrics are not Unique

- This is a general rule about the relationship between geometry and metrics:
- the metric certainly encodes the geometry of the space it describes, but not generally in a unique way or even obvious way.
- There can be many metrics that can describe the same space, but they must all have the intrinsic geometry of that space.
- Again, the fact that $g_{\mu\nu}$ depends on r (on position) does not guarantee that the space is curved.

A Two-Way Street

- As a practical matter, you can get the metric if you know the squared distance element of your space (as we have done here);
- or you can work in reverse: If you know the metric of your space, you can calculate the squared distance element of your space.
- Both calculations are important and common.

Major Topic: Parallel Transport of Vectors

- One way to think about a curved surface that does not involve the metric is to consider how the space affects the **parallel transport of a vector**.
- The notion of parallel transport involves moving a vector around in the space while keeping it always parallel to itself; never rotating it.
- (Recall the idea of acceleration $\Longleftrightarrow$ changing velocity)

Parallel Transport of Vectors: Flat Space

If you imagine performing this operation on some vector $\vec{V}$ in a flat space, then neither the vector nor its components will change at all as it is moved around. If its transport is a closed path, it will be exactly the same at the end of its "journey" as when it left.

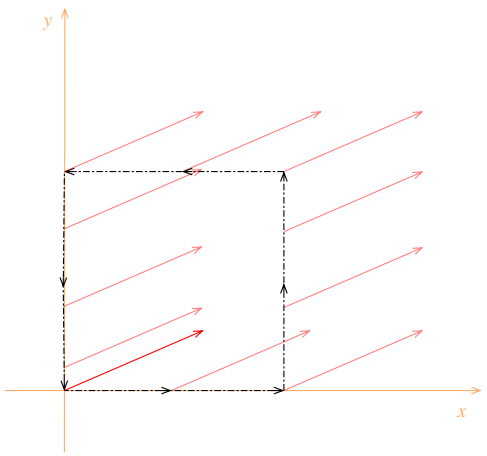

Figure: Transporting the red vector counterclockwise in the (x, y) plane. Note that its final direction is parallel to its initial direction.

Mathematically,

$$\vec{V}_f - \vec{V}_i = 0$$

Parallel Transport of Vectors: Spherical Space

We get a very different result if we do this same operation on the 2-d surface of a 3-sphere. Although we never allow the vector to rotate, it nevertheless ends up rotated through some angle measured from its initial orientation.

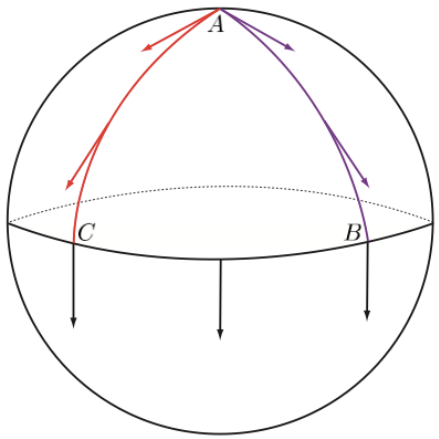

Figure: Transporting a vector along a closed path on the surface of a sphere. Note that its final direction is *not* parallel to its initial direction.

Mathematics of Transporting Vectors

Imagine the vector moves from some initial position (I) to some final position (F) along a path s and see how the vector changes. We could write this as:

$$\begin{aligned}\delta \vec{V} &= \vec{V}_F - \vec{V}_I \\ &= \int_I^F \frac{\partial \vec{V}}{\partial s} ds\end{aligned}$$

In a flat space, this is simply

$$\delta \vec{V} = \int_I^F \frac{d \vec{V}}{ds} ds \tag{10}$$

Mathematics of Transporting Vectors

But in a curved space, the above expression is incomplete because the coordinate basis vectors $\hat{e}_\mu$ themselves can change as we move the vector. To correctly write the derivative of a vector we must take this change into account:

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\mu}{\partial x^\beta}\hat{e}_\mu + V^\mu \frac{\partial \hat{e}_\mu}{\partial x^\beta} \tag{11}$$

(The Einstein summation convention is in force from here on out.)

The first term on the right-hand side of this equation reflects the change in the vector components V^μ itself, and the second term represents the change in the basis vectors $\hat{e}_\mu$ of the coordinates.

Mathematics of Transporting Vectors

This derivative is generally written in a slightly different way. If we write

$$\frac{\partial \hat{e}_\mu}{\partial x^\nu} = \Gamma^\alpha_{\mu\nu} \hat{e}_\alpha \tag{12}$$

then we can rewrite Eq. 11 as

$$\begin{aligned} \frac{\partial \vec{V}}{\partial x^\beta} &= \frac{\partial V^\alpha}{\partial x^\beta} \hat{e}_\alpha + V^\mu \Gamma^\alpha_{\mu\beta} \hat{e}_\alpha \\ &= \left(\frac{\partial V^\alpha}{\partial x^\beta} + V^\mu \Gamma^\alpha_{\mu\beta} \right) \hat{e}_\alpha \end{aligned} \tag{13}$$

The term in the parenthesis takes into account both the change in the vector components as we move around in the space, and also the change in the coordinate basis vectors.

It is called the **covariant derivative** of the vector $\vec{V}$.

Christoffel Symbols

- The terms $\Gamma^{\alpha}_{\mu\beta}$ are called **Christoffel symbols**: **They take into account how the coordinate basis vectors change as we move around in the space.**
- The way to think of them is that Γ^{i}_{jk} is the ith component of the change of [the jth basis vector with respect to the kth coordinate].
- In a *flat* space with constant basis vectors, the Christoffel symbols are zero.
- In curvilinear coordinate systems they are *not* zero, even if the space described is flat.
- For intrinsically curved spaces they are not zero either.

Parallel Transport

To parallel transport a vector means to transport it in such a way that its covariant derivative vanishes. Mathematically we may write

$$\frac{\partial V^\alpha}{\partial x^\beta} + V^\mu \Gamma^\alpha_{\mu\beta} = 0 \tag{14}$$

Parallel Transport (cont.)

- Again: Parallel transport does *not* mean that the vector will necessarily end up parallel to its initial orientation. That will only be true in a flat space.
- In more general spaces the notion that a vector in one part of the space is parallel to a vector in another part of the space might not even make sense. (For instance, what does it mean to say that two displacement vectors along Earth's surface, one at the equator and the other at the south pole, are parallel to each other?)
- In a curved space, the closest we can get to "parallel" is "parallel *transport*."

Parallel Transport (cont.)

- So getting back to the parallel transport of the vector $\vec{V}$, what we would like to know is how the vector changes as we parallel transport it around the space.
- Specifically, we would like to know how the vector changes if we parallel transport it around a closed path where our final point coincides with our starting point.
- Reason? Knowing this tells us about the geometry of our space!

Parallel Transport (cont.)

We imagine that the path of the vector is made up of short segments along two independent coordinates of our space, call them x^1 and x^2 , something like the situation shown in Fig. 5, with the path shown in blue:

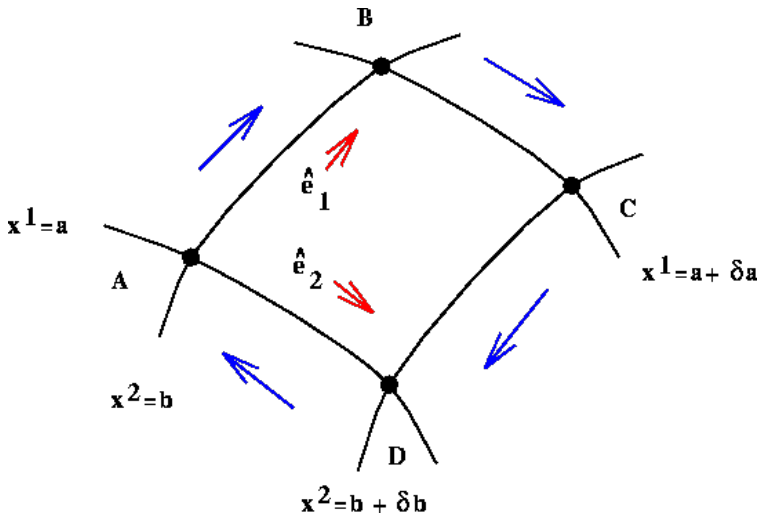

Figure: A possible path around which to parallel transport a vector.

We let $x^1 \in (a, a+\delta a), x^2 \in (b, b+\delta b)$

Calculate the Change of the Vector

The change in our vector along two points on our path, say along a constant value $x^1 = a$ is

$$\delta V^\alpha_{x^1=a} = \int_{x^1=a} \frac{\partial V^\alpha}{\partial x^2} dx^2 \tag{15}$$

Because we parallel transport this vector, we know that its covariant derivative is zero. That means we can replace the integrand using Eq. 14. Substituting gives

$$\delta V^\alpha_{x^1=a} = -\int_{x^1=a} \Gamma^\alpha_{\nu 2} V^\nu dx^2 \tag{16}$$

Summing Over Path Segments

We have four segments that must be traversed to bring the vector back to its starting point. After summing them all up, the result is

$$\begin{aligned} V^{\alpha}_{final} - V^{\alpha}_{initial} &= \delta b \delta a \left[\frac{\partial}{\partial x^{\lambda}} \Gamma^{\alpha}_{\nu\sigma} - \frac{\partial}{\partial x^{\sigma}} \Gamma^{\alpha}_{\nu\lambda} + \Gamma^{\alpha}_{\nu\lambda} \Gamma^{\nu}_{\mu\sigma} - \Gamma^{\alpha}_{\nu\sigma} \Gamma^{\nu}_{\mu\lambda} \right] V^{\nu} \\ &= R^{\alpha}_{\nu\sigma\lambda} V^{\nu} \delta b \delta a \end{aligned}$$

The term in the square brackets is called the **Riemann curvature tensor**, to which we give the symbol $\mathbf{R}$.

Riemann Curvature Tensor

So we have

$$R^{\alpha}_{\nu\sigma\lambda} \equiv \frac{\partial}{\partial x^{\lambda}}\Gamma^{\alpha}_{\nu\sigma} - \frac{\partial}{\partial x^{\sigma}}\Gamma^{\alpha}_{\nu\lambda} + \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\nu}_{\mu\sigma} - \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\nu}_{\mu\lambda} \tag{17}$$

- Now, $\mathbf{R}$ depends on the Christoffel symbols, and on their derivatives, which means it depends upon the first and second derivatives of the coordinate basis vectors.
- In a flat space $\mathbf{R}$ will be zero, though individual Christoffel symbols and their derivatives might not.
- In fact, the *condition* for a curved space is $\mathbf{R} \neq 0$.

Dimensionality versus Rank

- Note that the Riemann tensor is a rank 4 tensor.
- In regular three-space it will be a 3-dimensional tensor, with each of its indices running from 1 to 3. In a 2-dimensional space, like the surface of a sphere, it is 2-dimensional, with each index running from 1 to 2. In spacetime the Riemann tensor is a 4-dimensional tensor (1 time, 3 space). Its indices take on values from 0 to 3.
- *In all these spaces it is fourth rank*. Remember that a tensor's rank is how many indices it carries.
- Its dimensionality, on the other hand, matches the dimensionality of the space in which it lives.

$\mathbf{R}$ and the Metric

The importance of the Riemann curvature tensor is its connection to the metric tensor of a space. Both must have the same *dimensionality*, but they do not have the same *rank*. The metric is rank 2, not rank 4. It can be shown that

$$R^{\alpha}_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left(\frac{\partial^2 g_{\sigma\nu}}{\partial x^{\beta} \partial x^{\mu}} - \frac{\partial^2 g_{\sigma\mu}}{\partial x^{\beta} \partial x^{\nu}} + \frac{\partial^2 g_{\beta\mu}}{\partial x^{\sigma} \partial x^{\nu}} - \frac{\partial^2 g_{\beta\nu}}{\partial x^{\sigma} \partial x^{\mu}} \right) \tag{18}$$

This notation is a bit cumbersome, so we will instead use a compact form that emphasizes the tensor nature of $\mathbf{R} = R^{\alpha}_{\beta\mu\nu}$ and allows us to see its internal symmetries more easily.

Symmetries of $\mathbf{R}$

Let's define the **comma notation** for differentiating a tensor with respect to its various indices. For an arbitrary tensor $\mathbf{F}$,

$$F_{\alpha,\beta} \equiv \frac{\partial F_\alpha}{\partial x^\beta} \tag{19}$$

so the comma in the subscript implies differentiation. Using this shorthand notation, Eq. 18 is

$$R^\alpha_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left(g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu} \right) \tag{20}$$

Now we can easily see that $R^\alpha_{\beta\mu\nu}$ is antisymmetric under exchange of its first and third indices, as well as its second and fourth. In addition, it is symmetric under exchange of its first pair of indices with its second pair.

Given its symmetries, $\mathbf{R}$ is much less complicated than it looks: in four dimensions the number of independent components of $\mathbf{R}$ is "only" 20.

Summary

To put all of this into perspective, it is good to remember that our definition of curvature depends upon the idea of parallel transport. This is key: we know that

- **in a flat space a vector will be unchanged if moved around in the space while being kept parallel**.

On the other hand,

- **in a curved space there is generally no way to move a vector around and keep it parallel; it will be modified as it moves around**, even if we keep it "locally parallel."

This modification will happen in such a way that

- **if we bring the vector back to where we started it will no longer be the same vector we had when we started.**

Summary (cont.)

- Our ability to deal with this concept of parallel transport depended on taking the **covariant derivative** of a vector, a derivative that takes into consideration not only changes of the vector components, but **also changes to the coordinate basis vectors**.
- We found that in a general coordinate system, in which the basis vectors are not fixed, it is their turns and twists that describe the curvature of a space.

Summary (cont.)

- Of course, if we choose a small enough region, then we can approximate the space as flat, at least to some degree.
- This is similar to approximating a region in the world line of an accelerating object in flat space with a momentarily corresponding inertial frame.
- Just as in that case, we can approximate any point in space (assuming it is continuous and differentiable) with a locally flat tangent plane.

Summary (cont.)

- That does not, however, change the globally curved nature of the space, and the Riemann tensor picks out this curvature for us.
- Specifically–and as we will see later–the Riemann tensor in 4-dimensional spacetime allows us to connect the *curvature* of spacetime to *gravity*.